M.Sc. (P) Mathematics

Roll No. .... Total Pages : 3

MDE/M-16 **4028**

ADVANCED ABSTRACT ALGEBRA

Paper : I, MM-401

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *five* questions in all, selecting at least *one* question from each section. All questions carry equal marks.

SECTION–I

1. (a) Prove that a group of order p^n has a composition series of length n. Also write down a composition series for the symmetric group S_4.

(b) Let A and B be subgroups of a group G such that $G = \langle A, B \rangle$. Prove that $[A, B] \Delta G$.

2. (a) Prove that subgroup and factor group of a nilpotent group are again nilpotent.

(b) Let A and B be normal subgroups of a group G such that both G/A and G/B are solvable. Prove that $G/(A \cap B)$ is solvable.

SECTION–II

3. (a) Let $F \subseteq K \subseteq L$ be fields. Prove that if $[K : F] < \infty$ and $[L : K] < \infty$, then $[L : F] < \infty$ and $[L : F] = [L : K] [K : F]$.

(b) Prove that if $\alpha, \beta \in K/F$ are algebraic of coprime degrees, then $[F(\alpha, \beta) : F] = [F(\alpha) : F] [F(\beta) : F]$.

4. (a) Find the degree of the splitting field of $(X^2 + 1)(X^4 - 2)$ over Q.

(b) Let F be a field such that $F^* = F\{0\}$ is a cyclic group (under multiplication). Prove that F is a finite field.

5. (a) Find the Galois group of $f(X) = (X^2 - 1)(X^3 - 2)$ over Q.

(b) Prove that the polynomial $X^5 - 9X + 3$ is not solvable by radicals over Q.

SECTION–III

6. (a) Let $T \in A_F(V)$ has all characteristic roots in F. Prove that T satisfies a polynomial of degree $n = \dim_F(V)$ over F.

(b) Let $T \in A_F(V)$ be such that $T^2 = T$, $\dim_F(V) = n$. Prove that T is diagonalizable.

7. (a) Prove that two nilpotent linear transformations on V are similar iff they have the same invariainty.

(b) Prove that if $p(X) \in F[X]$ is the minimal polynomial of $T \in A(V)$ and V is a cyclic module relative to T, then $\exists$ a basis of V in which the matrix of T is $C(p(X))$.

SECTION–IV

8. (a) Prove that if R is a ring with unity, then $\mathrm{Hom}_R(R_R, R_R) \cong R$ as rings.

(b) Prove that submodule of a completely reducible R-module M is a direct summand of M.

9. (a) Prove that every nil left ideal of a left Artinian ring is nilpotent.

(b) Let a, b be elements of a Noetherian ring R with unity such that $ab = 1$. Prove that $ba = 1$.

10. (a) Prove that if I is a minimal left ideal of a ring R, then either $I^2 = \{0\}$ or I is generated by an idempotent.

(b) Find the Abelian group generated by a and b, where $2a = 3b$.
