M.Sc (P) Math

Roll No. ....

Total Pages : 03

DMDE/M-18 4028

ADVANCED ABSTRACT ALGEBRA

MM-401

Time : Three Hours] [Maximum Marks : 80

Note : Attempt *Five* questions in all, selecting at least *one* question from each Unit. All questions carry equal marks.

Section I

1. (a) Prove that a group of order p^n (P is a prime and $n \geq 0$) has a composition series of length n.

   (b) Let A and B be subgroups of a group G such that $[A, B, B] = \{e\}$. Prove that [A, B] is a Abelian subgroup of G.

2. (a) Prove that subgroup and a factor group of a nilpotent group are again nilpotent.

   (b) Define the derived series $\{\delta_i(G)\} i \geq 0$ of a group G and prove that G is soluble iff $\exists$ an integer $s \geq 0$ such that $\delta_k(G) = \{e\}$.

Section II

3. (a) Prove that if P is a prime of field, then either $P \cong Q$ or $P = z/pz$.

   (b) Find the degree of the splitting field of $(X^2 - 1)(X^2 - 2)(X^2 - 3)$ over Q.

4. (a) Prove that if $F^* = F'\{0\}$ is a cyclic group, then F is a finite field.

   (b) Prove that a finite normal extension is the splitting field of some polynomial.

5. (a) Find the Galois group of $X^4 - 2$ over Q.

   (b) Prove that the Galois group of $X^n - a \in F[X]$ is soluble.

Section III

6. (a) Prove that if $T \in A(V)$ has all its ch. roots in F, then there is a basis of V in which the matrix of T is triangular.

   (b) Prove that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ is nilpotent and find its invariants.

7. (a) Prove that invariants of a nilpotent transformation are unique.

   (b) Let K be an entension of F. Prove that if $A, B \in F_n = M_n(F)$ are similar in K_n, then they are already similar in F_n.

Section IV

8. (a) Let I be a left ideal of a ring R with unity. Prove that I is a direct summand of R iff I is generated by an idempotent.

   (b) Let A and B be submodels of R-modules M and N respectively prove that :

$$\frac{M \times N}{A \times B} \cong \frac{M}{A} \times \frac{N}{B}$$

9. (a) Prove that in a Noetherian ring every nil ideal is nilpotent.

   (b) Let N be a submodule of an R-module M such that both N and M/N are Artinian R-module. Prove that M is an Artinian R-module.

10. (a) Prove that a submodule of a free module of rank n over a PID is again free of rank $\leq n$.

    (b) Prove that the Abelian group G generated by x, y such that $2x = 3y$ is cyclic.